is important an analysis of second derivative.

Second Derivative Test: Suppose f is differentiable at $\bar{p}$ and $\bar{p}$ is a critical point of f.

1.) If $f_{xx}(\bar{p}) > 0$ and $D(\bar{p}) = f_{xx}(\bar{p}) f_{yy}(\bar{p}) - (f_{xy}(\bar{p}))^2 > 0$. then f has a local min at $\bar{p}$.

2) If $f_{xx}(\bar{p}) < 0$ and $D(\bar{p}) = f_{xx}(\bar{p}) f_{yy}(\bar{p}) - (f_{xy}(\bar{p}))^2 > 0$ then f has a local max at $\bar{p}$.

3.) If $D(\bar{p}) = f_{xx}(\bar{p}) f_{yy}(\bar{p}) - (f_{xy}(\bar{p}))^2 < 0$, then f has a saddle point at $\bar{p}$.

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LAST TIME 2nd Derivative Test

$D = f_{xx} f_{yy} - (f_{xy})^2$, $\bar{P}$ a CP.

1. $f_{xx}(\bar{p}) > 0$ and $D(\bar{p}) > 0 \Rightarrow \bar{p}$ a local min point

2. $f_{xx}(\bar{p}) < 0$ and $D(\bar{p}) > 0 \Rightarrow \bar{p}$ a local max point

3. $D(\bar{p}) < 0 \Rightarrow \bar{P}$ is a saddle point.

Note.

① If $D(\bar{p}) = 0$, we cannot say anything about $\bar{p}$ with this test.

② You could use $f_{yy}(\bar{p})$ in place of $f_{xx}(\bar{p})$

Ex. Classify the critical points of $f(x,y) = xy + e^{-xy}$ with the 2nd derivative test.

Sol. $\nabla f = \langle y - ye^{-xy}, x - xe^{-xy} \rangle$

$= \langle y(1-e^{-xy}), x(1-e^{-xy}) \rangle$

$\therefore \nabla f = \bar{0}$ iff $\begin{cases} y(1-e^{-xy}) = 0 \\ x(1-e^{-xy}) = 0 \end{cases}$

iff $\begin{cases} y=0 \text{ or } 1-e^{-xy}=0 \\ x=0 \text{ or } 1-e^{-xy}=0 \end{cases}$

Note $e^{-xy} = 1$ iff $-xy = 0$ iff $x=0$ or $y=0$

$\therefore \nabla f = \bar{0}$ iff either $x=0$ or $y=0$

↳ because $x=0$ or $y=0$ implies both of the conditions for $\nabla f = \bar{0}$

~~Picture of CP~~

Now we need $D(x,y)$:

$f_{xx} = y^2 e^{-xy}$ $\quad f_{yy} = x^2 e^{-xy}$

$f_{xy} = f_{yx} = 1 - (e^{-xy} - xye^{-xy}) = 1 - e^{-xy}(1-xy)$

$\therefore D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

$= (y^2 e^{-xy})(x^2 e^{-xy}) - (1 - e^{-xy}(1-xy))^2$

$D(x,y) = 0$ uniformly when $x=0$ or $y=0$. (Verify)

$\therefore$ Nothing can be said with 2nd derivative test.

Ex. Classify CPs of $f(x,y) = x^2+y^2+xy+y$ with 2nd derivative test.

Sol: $\nabla f = \langle 2x+y,\ 2y+x+1 \rangle$

$\therefore\ \nabla f = \bar{0}$ iff $\begin{cases} 2x+y=0 \\ 2y+x+1=0 \end{cases}$ iff $\begin{cases} y=-2x \\ 2(-2x)+x+1=0 \end{cases}$

iff $\begin{cases} -3x+1=0 \\ y=-2x \end{cases}$ iff $\begin{cases} x=\frac{1}{3} \\ y=-\frac{2}{3} \end{cases}$

$\therefore$ We have a unique CP at $(\frac{1}{3}, \frac{-2}{3})$.

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$ ~~Wher~~ $f_{xx} = 2$, $f_{yy}=2$, $f_{xy} + f_{yx} = 1$

$\therefore\ D(x,y) = 2\cdot 2 - 1^2 = 3 > 0$

$\therefore$ Because $f_{xx}(\frac{1}{3}, \frac{-2}{3}) = 2 > 0$ and $D(\frac{1}{3}, \frac{-2}{3}) = 3 > 0$ we have by 2nd derivative test: $(\frac{1}{3}, \frac{-2}{3})$ is a local min point.

With local min ~~point~~ value $f(\frac{1}{3}, \frac{-2}{3}) = \left(\frac{1}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right) + \left(\frac{-2}{3}\right)$

$= \frac{1}{9} + \frac{4}{9} - \frac{2}{9} - \frac{6}{9} = -\frac{1}{3}$

Ex: Classify CPs of $f(x,y) = x^3+y^3-3x^2-3y^2-4x$.

Sol: CPs are given when $\nabla f = \bar{0}$:

$\nabla f = \langle 3x^2-6x-9,\ 3y^2-6y \rangle$

$= 3\langle x^2-2x-3,\ y^2-2y \rangle$

$= 3\langle (x-3)(x+1),\ y(y-2) \rangle$

$\therefore\ \nabla f = \bar{0}$ iff $\begin{cases} (x-3)(x+1)=0 \\ y(y-2)=0 \end{cases}$ iff $\begin{cases} x=3 \text{ or } y=-1 \\ y=0 \text{ or } y=2 \end{cases}$

	$x=3$	$x=-1$
$y=0$	$(3,0)$	$(-1,0)$
$y=2$	$(3,2)$	$(-1,2)$

$\therefore$ We have CPs: $(3,0), (-1,0), (3,2), (-1,2)$.

Next we compute $D(x,y)$:

$f_{xx} = 6x-6$, $f_{yy} = 6y-6$, $f_{xy} = f_{yx} = 0$

$\therefore\ D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 6(x-1)\cdot 6(y-1) - 0^2 = 36(x-1)(y-1)$

Now analyze P_s:

For $(3,0)$: $f_{xx}(3,0) = 6(3-1) > 0$ and $D(3,0) = 36(3-1)(0-1) < 0$

Because $D(3,0) < 0$, $(3,0)$ is a saddle point of f.

For $(-1,0)$: $f_{xx}(-1,0) = 6(-1-1) < 0$ and $D(-1,0) = 36(-1,-1)(0-1) > 0$

$(-1,0)$ is a local max point of f with local max value $f(-1,0) = 5$

For $(3,2)$: $f_{xx}(3,2) = 6(3-1) > 0$ and $D(3,2) = 36(3-1)(2-1) > 0$

$(3,2)$ is a local min point of f with local min value $f(3,2) = -31$

For $(-1,2)$: $f_{xx}(-1,2) = 6(-1-1) < 0$ and $D(-1,2) = 36(-1-1)(2-1) < 0$

Because $D(-1,2) < 0$, $(-1,2)$ is a saddle point of f.

Section ?: Lagrange Multipliers

IDEA: Method for solving constrained optimization problems.

Observation: If $f: \mathbb{R}^n \to \mathbb{R}$ has a level curve

$f(\bar{x}) = C$, then for every point $\bar{p}$ on this level curve: $\nabla f(\bar{p}) \cdot \nabla c = 0$

i.e ∇f is orthagonal to level curves.

IDEA: Set up Contrained optimization to be optimizations on a level curve...

In general: $\begin{cases} \text{Optimize } f(\bar{x}) \\ \text{Subject to } \underbrace{g_1(\bar{x}) = 0,\ g_2(\bar{x}) = 0, \ldots}_{\text{0-level curves}} \end{cases}$

Can be turned into optimize:

$F(\bar{x}, \lambda_1, \lambda_2, \ldots, \lambda_K) = f(\bar{x}) - \lambda_1 g_1(\bar{x}) - \lambda_2 g_2(\bar{x}) - \cdots - \lambda_K g_K(\bar{x})$

So $F(\bar{x}, \bar{\lambda})$ has CPs with $\nabla F = 0$

So Supposing the solution set of $g_1(\bar{x}) = 0 = g_2(\bar{x}) = \cdots = g_K(\bar{x})$ is closed and bounded, the local extreme of F determine global extreme of f under $g_1 = g_2 = \cdots = g_K = 0$

Now let's see this in Practice...

Ex: Optimize $f(x,y) = xe^y$ subject to $x^2+y^2=2$

Sol: Via Lagrange Multipliers:

$F(x,y,\lambda) = f(x,y) - \lambda\, g(x,y)$

with $g(x,y) = x^2+y^2-2$ (because $g(x,y)=0$ iff $x^2+y^2=2$)

So $F(x,y,\lambda) = xe^y - \lambda(x^2+y^2-2)$

$\therefore \nabla F = \langle e^y - \lambda(2x),\ xe^y - \lambda(2y),\ -(x^2+y^2-2)\rangle$

So $\nabla F = 0$ iff $\begin{cases} e^y - \lambda 2x = 0 \\ xe^y - \lambda 2y = 0 \\ -(x^2+y^2-2) = 0 \end{cases}$ iff $\begin{cases} 2\lambda x = e^y & (1) \\ 2\lambda y = xe^y & (2) \\ x^2+y^2 = 2 & (3) \end{cases}$

(Idea: Solve this system)

Now $2\lambda x = e^y$ implies $\lambda \neq 0$ (lest $e^y = 0$ which is nonsense)

~~$\therefore x = \frac{e^y}{2\lambda}$, So using equation 2: $2\lambda y = \left(\frac{e^y}{2\lambda}\right)e^y$~~

with equation (1) and (2) we obtain $2\lambda y = x(2\lambda x)$

$\therefore 2\lambda y = 2\lambda x^2$. Now $\lambda \neq 0$ yields $y = x^2$ So (3) becomes $x^2 + (x^2)^2 = 2$

ie. $(x^2)^2 + x^2 - 2 = 0$

ie. $(x^2+2)(x^2-1) = 0$

ie. $(x^2+2)(x-1)(x+1) = 0$

Negative Discriminant $\therefore x = 1$ or $x = -1$

Note λ does not matter for $f(x,y)$ as long as there is one and is constant.

$\therefore x=-1$: then $y = (-1)^2$ and $\lambda = \frac{e^y}{2x} = \frac{e^1}{2(-1)} = -\frac{e}{2}$

So $(-1,1)$ is a potential extreme point of f subject to $g=0$

$f(-1,1) = (-1)e^1 = -e$

$x=1$: $y = 1^2 = 1$, $\lambda = \frac{e^1}{2\cdot 1} = \frac{e}{2}$ $\therefore (1,1)$ is a potential extreme point of f subject to g, and $f(1,1) = 1e^1 = e$.

$\therefore -e$ is the global min and e is the global max of f subject to $x^2+y^2=2$.